LOGS AGAIN

Flushed with success at demystifying decibels, Joules Watt has not yet finished with logarithms...

JOULES WATT

A short while ago I commented upon what might be called slovenliness, regarding the use of decibels [1].

If not slovenliness, at least loose thinking about such matters causes confusion for the poor student. This has become true for novice radio amateurs as well. One confided to me, "It struck a little fear into nearly all the RAE class, when the mysterious dB arose in transmitter power discussions". I hope my short discourse earlier helped dispel some of the mystery.

But we are not out of the wood yet, because two colleagues took me to task with criticisms. One said, "You got no further towards helping anyone see the meaning of response curves. Are the 6 dB per octave slopes a drop-off in power gain or voltage gain?" The other fellow suggested I too was being a little slovenly, in that doubling the power is not exactly 3 dB up, therefore four times is not exactly 6 dB, and so on. He is right, of course, but the differences are very small, as Table 1 shows.

Much of all this argument appears to rest on the properties of logarithms. Long gone are the days when we slaved away looking up columns of figures in tables for something called the mantissa and something else called the characteristic – or are they? Nevertheless, it was only when we arrived at the integral calculus that we discovered a mysterious "irrational" number called e. As with π before it, we were told that "it has an unending, never repeating decimal part, which arises naturally..."

It soon became obvious that logs arise from this number e, as well as via the laws of indices generally. A logarithmic result to the base e is obtained when the area under a rectangular hyperbola is sought by using the integral

$$y=\int\frac{dx}{x}=\ln x \qquad (1)$$

The symbol "ln" reads "The natural logarithm of..." and is also written "$\log_e$". The common logs to the base ten (written $\log_{10}$, or just log for short from now on), cropped up recently in the discussion of decibels[1]. Log to the base 2 ($\log_2$) is seen in discussions concerning information and communications theory.

When physical quantities, instead of pure numbers, become involved in this kind of calculation, trouble tends to arise. Pure mathematicians avoid all that by claiming that they work with pure numbers only. Dimensions never come into things. Engineers, on the other hand, are always talking about so many amps, volts, newtons, watts and the question is asked as to whether the unit is inside the symbol – or attached separately. In other words, is the quantity R a number complete with dimensions ohms within it, or should it be written R ohms, where R is a pure number?

From the integral (1) y is certainly dimensionless, because if x has dimensions, the small bit of it, dx, has the same dimensions, and these cancel in the ratio. But if x does have dimensions, say watts, then the question is whether log of watts can equal a dimensionless number? This is not a real problem, because I have neglected the fundamental need for a constant to add on to any integration. In this case (for example) we can say "let x = p watts when y = 0" so that the constant is $-\ln p$ and the complete answer is:

$$y=\ln\frac{x}{p}$$

and everyone is happy, as we have a pure ratio of watts over watts again.

But this is not so for the 'quantity-calculus' people [2,3,4], who argue that the symbols of the units can be handled and juggled just like numbers. You can square them, divide, take logs of them. And so, as you might expect, a fairly hot controversy has raged over the claim to do this.

After considering these arguments, I incline towards the old view that quantities cannot have logs taken of them – or the sine, tangent and so on, for that matter. We can only use *ratios* of same-dimensioned quantities in the arguments of the transcendental functions. Of course, all the modernists will label my stand as reactionary.

In practice there is no trouble, because the measure of quantities is really a ratio in every case. A voltmeter reading 20 volts is really saying, "The basic unit (volt) goes into the value being read (20 volts) 20 times. i.e., 20 volts/volt equals 20 times." Therefore if we take logs of a power in watts, we are really taking cognizance of a number of times a quantity is in a ratio to the basic unit. Therefore 10log P is really 10log P(watts)/watt or so many dBW.

If we are told that the comparator unit is milliwatts, then the operation can be written

$$10\log\frac{P(\text{watts})}{\text{milliwatt}}=10\log\frac{10^3P(\text{milliwatts})}{\text{milliwatt}}$$

$$\therefore \qquad \text{dBm} = (30+\text{dBW})$$

TABLE 1

P_2/P_1	$10\log(P_2/P_1)$	Usually taken as:
2	3.0103	3dB
4	6.0206	6dB
8	9.0309	9dB
10	10	10dB
64	18.0618	18dB
100	20	20dB
2.749×10^{11} (30 doublings)	114.3914	114dB

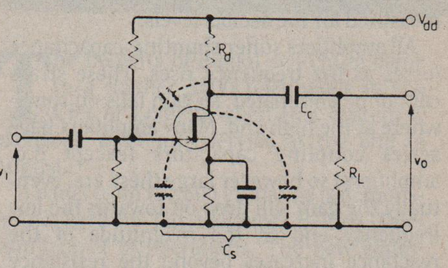


Fig.1. This typical j.fet resistance-capacitance coupled amplifier stage is typical of all such active device gain blocks, (with appropriate attention to input and output impedances). The unmarked components are assumed not to affect the frequency response. In other words, C_c dominates at the low end, while the strays can be lumped together as C_s to limit the performance at the high end.

THE PLOT THICKENS
(at least towards the high values..)

Displaying data logarithmically to expand the detail at the low end of the scale and compress it at the high end enables a huge range of information to be shown on one sheet of graph paper. This explains why log – log and log – linear graphs are so ubiquitous in engineering reports and scientific papers. You can get log paper in so many cycles (decades usually) along the two axes. One consequence is that the true origin goes off to infinity downwards and to the left along the axes. ($\log 0 \rightarrow \infty$).

Another reason why log plotting is so useful is that many phenomena vary logarithmically, which means many data measurements produce straight line plots on log – log paper. Other "laws" which would produce curves on linear-only paper, will produce straight line plots on log – lin paper.

Again, to overcome the "worry" about the meaning of units along the log axes, we think of the numbers as 'so many units per unit' of the quantity, so that you are never taking the log of anything but a pure ratio. (A ratio is a pure number, but a *rate* is a quotient of different quantities – like 'miles per hour'.)

IT BODES WELL

By measuring the power output of an amplifier and (with the same power meter) the power input, we can take their ratio and thus write down the power gain in dB. Whether this is the available power gain, or the transducer gain, or matched power gain and so on, is another confusion, as it dawns upon us that different people define power gain differently. Nevertheless, the ratio of two powers to give decibels is now basically simple.

The question might arise as to *relative* gain over, say, the frequency band of an amplifier. It might also be asked for in terms of the voltage gain. But we sigh with relief because the relative gain will be measured at the same point (the output terminals) and therefore across the same impedance, assumed to be a non-reactive load for instance. This means that voltages can be measured for the decibel levels.

All amplifiers suffer shunting-capacitance losses as the frequency rises. These strays cannot be eliminated, so gain falls off somewhere at the high end. Many amplifiers have series coupling capacitors (except d.c. amplifiers) so however large these are, eventually the gain will drop off towards the low frequency end as the magnitude of the reactance increases beyond the resistance values. In other words, all amplifiers are bandpass circuits. The bandwidth B is the frequency interval between the "3 dB down" points. Of course, specialized amplifiers might depart from this simple scenario. Baxandall tone control stages would do so, for example. Figure 1 shows how a simple voltage amplifier fet stage appears.

Figure 2 shows the well known equivalent generator circuit for such a stage incorporated into the mid-frequency range, where the capacitances have no effect; the low frequency region, where C_c dominates, but the shunt strays do not; and at the high frequency end where the total shunt capacitance C_s dominates, but C_c does not. These notional independent regions are the usual assumptions made about a fairly wide-band amplifier, but for narrow-band cases other methods have to be used.

The mid-band voltage gain is simply the voltage-controlled generator current multiplied by the total equivalent resistance at the output:

$$A_{mid}=\frac{v_o}{v_i}=-g_mR_{eq}$$

where $R_{eq}=\dfrac{r_dR_dR_L}{r_dR_d+r_dR_L+R_dR_L}$

In order to see the effect of C_s at high frequencies, it is added in shunt to R_{eq} as shown in Fig.2(b)

$$A_h=\frac{v_o}{v_i}=-g_mZ$$

where Z is $\dfrac{-jR_{eq}X_s}{R_{eq}-jX_s}$ or $\dfrac{R_{eq}}{1+j\frac{R_{eq}}{X_s}}$

and $X_s=\dfrac{1}{\omega C_s}$

But $-g_mR_{eq}$ is A_{mid}

therefore: $\dfrac{A_h}{A_{mid}}=\dfrac{1}{1+j\frac{R_{eq}}{X_s}}$

This can be written $\dfrac{1}{1+j\frac{\omega}{\omega_2}}$

where $\omega_2=\dfrac{1}{R_{eq}C_s}$

The ratio of the high-frequency gain to the mid-frequency value, is called *normalization* – in this case, normalizing to the mid-frequency gain as a reference. It is as though we have set the mid value to unity, but normalization does more than that, it removes any dimensions of the quantities in the ratio, (although in this case, the As are already dimensionless).

The result derived for the normalization voltage gain is a complex number. You probably remember such numbers have real and imaginary parts. Because I introduced phase shifts resulting from the circuit reactances by using the operator j, the complex result arises naturally. It has the advantage that when written down in the polar form, the amplitude (or magnitude) and the phase angle are immediately given. In the present context, these two pieces of information tell us all we want to know about the amount of amplification on the one hand and the shift in phase of the output relative to the input, at any given frequency, on the other.

$$\therefore \frac{1}{1+j\frac{\omega}{\omega_1}}=\frac{1}{\sqrt{1+\frac{\omega^2}{\omega_2^2}}}\;\angle -\tan^{-1}\frac{\omega}{\omega_2} \qquad (2)$$

I have written this result using the rather out of fashion notation for amplitude A, and angle $\angle\,\Theta$, but which has merit and perhaps we should rehabilitate it a little.

Returning to Fig.2(c) where I show the equivalent circuit for the low-frequency end, the effect of C_s is now negligible, but that of C_c rises into prominence. The generator current divides into r_d and R_d in parallel as one path, and C_c in series with R_L as the other. I have combined the result of r_d and R_d in parallel as R_D and, using the current divider formula, the current through R_L can be written down. Knowing the current through the load resistor will give the voltage across it, namely, v_o.

$$\therefore \quad v_0=\frac{-R_Dg_mv_iR_L}{R_D+R_L-jX_c}$$

where $X_c=\dfrac{1}{\omega C_c}$

The algebra can be re-arranged to give a $1+j$ term in the denominator to make it look like the first result:

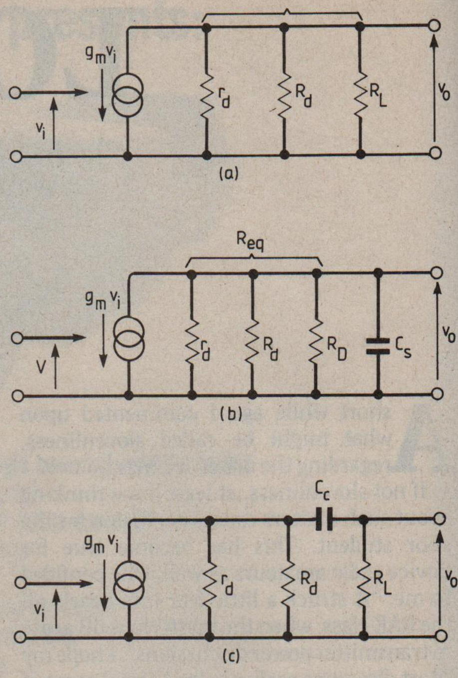


Fig. 2. In (a) the mid-band equivalent circuit is shown. The 'mid-band' can be defined as $\omega_{mid}=\sqrt{\omega_1\omega_2}$. At (b) is the effect of C_s which shunts the output circuit. The "3 dB down" frequency is that at which the magnitude of the reactance of C_s becomes equal to R_{eq}. This frequency, also known as the 'high-frequency cut off point', is denoted ω_2, where ω_2 is $2\pi f_2$. f_2 is the frequency in hertz.

The low-frequency case shown in (c) is a little more awkward, in that C_c is in series with R_L and this is in shunt across the rest of the resistive components. ω_1 is the cut off frequency at the low end and it occurs at the frequency where the reactance of C_c has a magnitude equal to R_L summed with the resultant of r_d and R_d in parallel.

$$\frac{v_0}{v_i}=\frac{-R_Dg_mR_L}{R_D+R_L-jX_c}=\frac{-R_{eq}g_m}{1+j\frac{R_D+R_L}{X_c}}\cdot\frac{j(R_D+R_L)}{X_c}$$

$$=\frac{-R_{eq}g_m\frac{\omega}{\omega_1}}{1+j\frac{\omega}{\omega_1}}$$

Where R_{eq} is the same as before.

Writing the final line for the normalized gain at low frequencies, we obtain:

$$\frac{A_L}{A_{mid}}=\frac{1}{1+j\frac{\omega}{\omega_1}}=\frac{\frac{\omega}{\omega_1}}{\sqrt{1+\frac{\omega^2}{\omega_1^2}}}\;\angle \pi-\tan\frac{\omega}{\omega_1}$$

This time ω_1 stands in for $1/(R_0+R_L)C_c$ and is another cut-off point – the low frequency one. These values, ω_1 and ω_2, are also called "break" frequencies. The reason for these terms becomes obvious when we get back to logs shortly.

Before taking logs of the amplitude of the low and high-frequency gain variations, I have plotted in Fig.3 the linear amplitude versus frequency result as an instructive illustration of a direct attack. This turns out to be less useful than at first thought. A log –

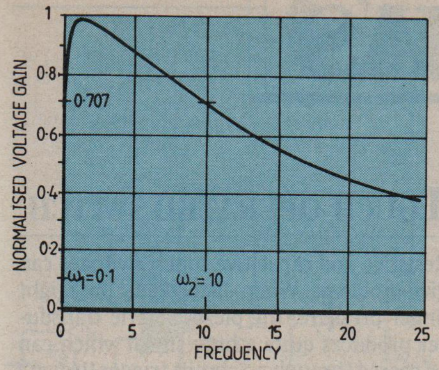


Fig.3. A very lop-sided result occurs if linear plots are attempted for frequency response. Also such curves accord very badly with subjective results, considering the logarithmic response of the ear.

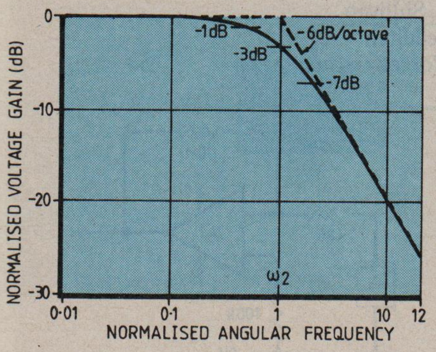


Fig.4. A Bode plot such as that shown here for the high-frequency response, is a very convenient way of using a logarithmic presentation to give a rapid overview of data quickly.

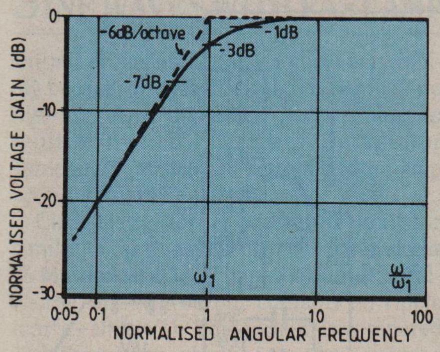


Fig.5. A similar plot at the low-frequency end of an amplifier's response gives similar information just as conveniently.

log plot (on log – lin paper, because decibels are used on the vertical axis) is much more fruitful. The log plots turn out to be "piecewise linear" and they are all associated with the name Hendrik Bode[5]. These Bode plots are a very quick way of seeing the frequency characteristics of an amplifier, once the complex gain equations have been derived and the logs taken.

Consider the high-frequency end of the amplifier response I have discussed above. We take common logs of the normalized gain expression (equation 2):

$$20\log\frac{A_h}{A_{mid}} = 20\log\frac{1}{\sqrt{1+\frac{\omega^2}{\omega_2^2}}} = -20\log\sqrt{\left(1+\frac{\omega^2}{\omega_2^2}\right)}$$

Taking twenty times the log is no mistake – as we now have the normalized voltage gain in decibels to plot against the log of the frequency.

If you take a look at the log equation above with $\omega << \omega_2$ then the right-hand side approaches log 1, which is 0. Therefore, on the log plot there is a horizontal line at 0 dB which corresponds to the normalized mid-band value. Now consider $\omega >> \omega_2$. The "1" can be neglected now in the bracket of the log argument:

$$\therefore \quad \left(\frac{A_h}{A_{mid}}\right)_{dB} = -20\log\frac{\omega}{\omega_2}$$

Plotting this on Fig.4 gives a straight line with a slope of −20 units for every factor of ten times ω increase. The units here, of course, are decibels. The two asymptotes have a common point at $\omega=\omega_2$. Therefore on the log plot, an approximation to the amplifier response has been obtained with a sharp "break" point at ω_2. You can see that in practice, at $\omega=\omega_2$, the argument of the logarithm is $\sqrt{2}$, and the actual response is down 3 dB at this point. When the frequency is at twice the cut off value, then the actual response is $20\log\sqrt{5} = 7$ dB down. This is one decibel lower than the asymptote at the $2\omega_2$ point. Similarly, at half ω_2 the response is again one dB lower. This means we can sketch the actual smooth response curve quite accurately. The last notable comment to make is that the slope of the response curve approaches "6 dB per octave". We now have all the 'jargon' commonly met in these discussions. The "3 dB down" point; "6 dB per octave" or "20 db per decade"; "break point" and so on.

This kind of response is typical of a "one-pole system", where the pole is at the frequency which makes the denominator zero in equation 6. The lower break point is obtained in the same way, except that there is a "zero" in the numerator (see equation 3) as well as a pole at ω_1. I will "leave it as an exercise for the student" as the saying goes, to sketch the Bode plot at the low end. You should get a curve such as that in Fig.5. I have also left out of the discussion the plot of the phase angle from the $\tan^{-1}$ information. For completeness this should be done and readers can do it themselves, if interested.

I have mentioned poles and zeros. They have arisen naturally in logarithmic discussions of response curves via Bode plots. But any further look at them is a whole topic, and suffice it must be for now to hope that I have answered my critic's point about "what does it mean about 3 dB down on response curves?" and in doing so might have pointed up for you a few of the mysteries of these curves...

References

1 'Joules Watt', 'Ringing the Changes on Bels' *Electronics Wireless World*, September 1986

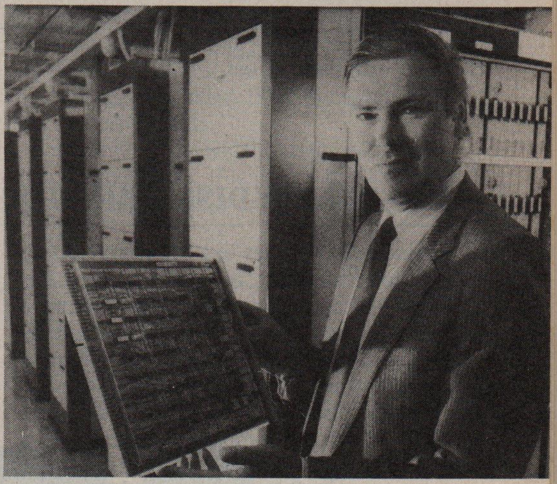
2 J.E. Boggs, 'The Logarithm of "Ten Apples"' *Jour, Chem. Edn. 35*, p30, January 1958

3 G.N. Copley, (Lett.) *Jour. Chem.Edn. 35*, p366, 1958

4 E.A. Guggenheim, 'Units and Dimensions' *Philosophical Magazine, 33*, p479, July 1942.

5 H.W. Bode "Network Analysis and Feedback Amplifier Design" Van Nostrand, 1957.

System X speedup



The benefits of digital switching came to a further 3500 telephones in the City of London when this System X exchange went into service at the end of August.

A panel from the 600 exchange is demonstrated here by Ian Vallance, British Telecom's chief of operations. The new equipment forms one of two exchanges housed at Wood Street and it replaces two floors of 39-year-old Strowger electromechanical switches. The other exchange, 726, will be commissioned shortly.

All telephone users should benefit from an improvement in the quality of service. But System X offers many new features to suitably-equipped customers, such as direct digital interfacing.

BT's System X programme has been lagging some 15 months behind schedule because of supply problems. But over 70 System X local exchanges are now in operation nationwide and new digital exchanges are entering service at the rate of one every working day. By the end of the decade, half the BT network and all its trunk circuits will be digital.

Even rural communities are gaining the facilities of System X with the introduction of the smaller UXD5 digital exchange: some 200 are now in use and a further 300 are being installed or are on order. But seven million subscribers will remain on analogue TXE4 systems.

The switch-over at Wood Street took place at lunch-time on a Friday, which is reckoned to be the quietest time in the City. With military precision a squad of exchange staff completed the conversion in a two-minute operation: tugging on handfuls of string festooning the distribution frame, they disconnected the old system by pulling out thousands of tiny plastic wedges. On a further command, with more string-pulling they switched in the new.